

L^p -Regularity of Fourier Integrals on Product Spaces

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Abstract

We give an extension of the classical L^p -regularity theorem of Fourier integral operators into product space: $\mathbb{R}^{N_1} \times \mathbb{R}^{N_2} \times \cdots \times \mathbb{R}^{N_n}$. We require that their phase functions satisfy the crucial non-degeneracy condition, on each coordinate subspace \mathbb{R}^{N_i} for $i = 1, 2, \dots, n$. In the other hand, by classifying on their symbols of order m , we prove that the Fourier integral operators are bounded on $L^p(\mathbb{R}^N)$, whenever

$$\left| \frac{1}{2} - \frac{1}{p} \right| \leq \frac{-m}{N - n}$$

for $N = N_1 + N_2 + \cdots + N_n > n$.

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1 Introduction

The classical L^p -regularity theorem of Fourier integral operators was established in 1991, by Seeger, Sogge and Stein in [1]. In this paper, we give an extension of the above result into product spaces.

Such *product theorem* was first investigated by R. Fefferman and Stein in [2] for translation invariant singular integrals. It was later refined by Nagel, Ricci and Stein in [6] which gives a classification between the corresponding symbols and kernels by Fourier transforms. A recent result for singular integrals that are non-translation invariant is proved in [7]. The class of

operators introduced there can be viewed as either singular integrals of non-convolution type, or as pseudo differential operators whose symbols satisfy certain characteristic properties. The regarding estimation developed in [7] will come in part of our L^p -estimate in the case of $N = n$, whereas the Fourier integral operators are essentially pseudo differential operators. A further background of Fourier integral operators can be found in the work of Sogge [4], Peral [8] and Treves [9].

The paper is organized as follow: We state the main result in section 2. In section 3, we recall the combinatorial estimates obtained in [7] for the desired Littlewood-Paley projections. In section 4, we give a local L^p -estimate of Fourier integral operators in the case of $N = n$, for which they are essentially pseudo differential operators. We give the L^2 -estimate in section 5. In section 6, we sketch the proof by an heuristic argument. The L^p -estimate will be completed in section 7, by estimating on the kernels.

2 Statement of Main Result

Let $N = N_1 + N_2 + \dots + N_n$ be the dimension of a product space, whereas each N_i is the homogeneous dimension of the i -th subspace, for $i = 1, 2, \dots, n$. We write

$$x = (x^1, x^2, \dots, x^n) \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2} \times \dots \times \mathbb{R}^{N_n} = \mathbb{R}^N \quad (2.1)$$

and $\xi = (\xi^1, \xi^2, \dots, \xi^n)$ denotes its dual variable in the frequency space. Define the inner product

$$x \cdot \xi = x^1 \cdot \xi^1 + x^2 \cdot \xi^2 + \dots + x^n \cdot \xi^n. \quad (2.2)$$

Let $f \in \mathcal{S}$ be a Schwartz function. A Fourier integral operator is defined by

$$(\mathcal{F}f)(x) = \int_{\mathbb{R}^N} e^{2\pi i \Phi(x, \xi)} \sigma(x, \xi) \widehat{f}(\xi) d\xi \quad (2.3)$$

where $\sigma(x, \xi) \in C^\infty(\mathbb{R}^N \times \mathbb{R}^N)$ has a compact support in x . The phase function Φ is real and is in the form of

$$\Phi(x, \xi) = \Phi_1(x^1, \xi^1) + \Phi_2(x^2, \xi^2) + \dots + \Phi_n(x^n, \xi^n). \quad (2.4)$$

Each Φ_i , $i = 1, 2, \dots, n$ is homogeneous of degree one in ξ^i , smooth away from the coordinate subspace $\xi^i = 0$, and satisfies the non-degeneracy condition

$$\det \left(\frac{\partial^2 \Phi_i}{\partial x_j^i \partial \xi_k^i} \right) (x^i, \xi^i) \neq 0 \quad (2.5)$$

for $\xi^i \neq 0$ on the support of $\sigma(x, \xi)$. By singular integral realization, we write

$$(\mathcal{F}f)(x) = \int_{\mathbb{R}^N} f(y) \Omega(x, y) dy \quad (2.6)$$

where

$$\Omega(x, y) = \int_{\mathbb{R}^N} e^{2\pi i (\Phi(x, \xi) - y \cdot \xi)} \sigma(x, \xi) d\xi. \quad (2.7)$$

By the *localization principal* of oscillatory integrals, Ω in (2. 7) has singularity appeared at

$$\nabla_\xi(\Phi(x, \xi) - y \cdot \xi) = 0 \quad (2. 8)$$

for some ξ . See chapter VIII of [3]. At each x , we consider the variety

$$\Sigma_x = \{y : y = \nabla_\xi \Phi(x, \xi) \text{ for some } \xi\} \quad (2. 9)$$

which is the locus of the singularity of $y \rightarrow \Omega(x, y)$. By definition of Φ in (2. 4)-(2. 5), $\nabla_\xi \Phi_i$ is homogeneous of degree zero in ξ^i , for $i = 1, 2, \dots, n$. The projection of Σ_x on each coordinate subspace \mathbb{R}^{N_i} has dimension at most equal to $N_i - 1$. We next introduce the symbol class which was first investigated in [7]:

Definition of S_ρ Let $0 \leq \rho < 1$. A symbol $\sigma(x, \xi)$ belongs to the symbol class S_ρ^m if it satisfies the differential inequality

$$\left| \frac{\partial^\alpha}{\partial \xi^\alpha} \frac{\partial^\beta}{\partial x^\beta} \sigma(x, \xi) \right| \leq A_{\alpha, \beta} \prod_{i=1}^n \left(\frac{1}{1 + |\xi^i| + |\xi|^\rho} \right)^{|\alpha^i|} (1 + |\xi|)^{m + \rho|\beta|} \quad (2. 10)$$

for every multi-index $\alpha = (\alpha^1, \alpha^2, \dots, \alpha^n)$ and β .

Suppose $m \leq 0$. For $\rho = 0$ in (2. 10), $\sigma(x, \cdot)$ is essentially a *product symbol* introduced in [6]. When $0 < \rho < 1$, differential inequality (2. 10) has a mixture of homogeneities. In general, S_ρ^0 forms a sub-class of the *exotic class* $S_{\rho, \rho}^0$, see chapter VII of [3]. In the other hand, $\sigma \in S_\rho^0$ satisfies a variant of Marcinkiewicz condition:

$$\left| \left(\xi \frac{\partial}{\partial \xi} \right)^\alpha \sigma(x, \xi) \right| \leq A_\alpha \quad (2. 11)$$

for every α uniformly in x . Our main result is the following:

Theorem 1 Let $\sigma \in S_\rho^m$. Fourier integral operator \mathcal{F} defined in (2. 3)-(2. 5) with $-(N-n)/2 < m \leq 0$, initially defined on \mathcal{S} , extends to a bounded operator on $L^p(\mathbb{R}^N)$, whenever

$$\left| \frac{1}{2} - \frac{1}{p} \right| \leq \frac{-m}{N-n} \quad (2. 12)$$

for $N - n > 0$.

As was discussed in 6.13, chapter IX of [3], estimate (2. 12) is sharp in the sense of the following: Suppose $n = 1$. Let $a(x) \in C_o^\infty(\mathbb{R}^N)$ not vanishing on $|x| = 1$ and $\eta(\xi) \sim (1 + |\xi|)^m$. Fourier integral operator \mathcal{F} , with phase function $\Phi(x, \xi) = x \cdot \xi \pm |\xi|$ and symbol $\sigma(x, \xi) = a(x)\eta(\xi)$, is not bounded on $L^p(\mathbb{R}^N)$ for $|1/2 - 1/p| > m/(1 - N)$.

We will carry out our L^p -estimate in the framework of Littlewood-Paley projections developed in [7]. In particular, the frequency space will be decomposed into the product of Dyadic balls. For each ball belonging to the coordinate subspace, we need to construct a second Dyadic decomposition, in analogue to the estimation given implicitly in [10]. In compare to the work down in [1], we have obtained a decaying estimate for the kernel of every partial sum operators. This eventually makes the breakthrough of our analysis.

Abbreviations:

- ◊ We write $\Phi_\xi = \nabla_\xi \Phi$ and $\Phi_{\xi_i} = \nabla_{\xi_i} \Phi_i$ for $i = 1, 2, \dots, n$.
- ◊ Unless otherwise indicated, we write $\int = \int_{\mathbb{R}^N}$ and $\mathbf{L}^p = \mathbf{L}^p(\mathbb{R}^N)$.
- ◊ We write $\mathbf{H}^1 = \mathbf{H}^1(\mathbb{R}^N)$ to be the \mathbf{H}^1 -Hardy space, and \mathbf{BMO} denotes the space of bounded mean oscillations.
- ◊ We always write A as a generic, positive constant with a subindex indicating its dependence.

3 Littlewood-Paley Projections

In this section, we recall some combinatorial estimates from [7], with respect to our desired Littlewood-Paley projections. These results will be in principal roles of our later estimation.

Let $t_i, i = 1, 2, \dots, n$ be positive integers. We write $q = 1/\rho$ where $0 \leq \rho < 1$. Consider the n -tuples

$$\mathbf{t}_i = (2^{-qt_i}, \dots, 2^{-qt_i}, 2^{-t_i}, 2^{-qt_i}, \dots, 2^{-qt_i}) \quad (3.1)$$

where 2^{-t_i} is located on the i -th component, for every $i = 1, 2, \dots, n$. We thus define the non-isotropic dilations

$$\begin{aligned} \mathbf{t}_i \xi &= (2^{-qt_i} \xi^1, \dots, 2^{-qt_i} \xi^{i-1}, 2^{-t_i} \xi^i, 2^{-qt_i} \xi^{i+1}, \dots, 2^{-qt_i} \xi^n) \\ i &= 1, 2, \dots, n. \end{aligned} \quad (3.2)$$

In the other hand, we simultaneously define

$$\mathbf{t} = (2^{-t_1}, 2^{-t_2}, \dots, 2^{-t_n}), \quad \mathbf{t}^{-1} = (2^{t_1}, 2^{t_2}, \dots, 2^{t_n}) \quad (3.3)$$

and

$$\mathbf{t} \xi = (2^{-t_1} \xi^1, 2^{-t_2} \xi^2, \dots, 2^{-t_n} \xi^n), \quad \mathbf{t}^{-1} \xi = (2^{t_1} \xi^1, 2^{t_2} \xi^2, \dots, 2^{t_n} \xi^n). \quad (3.4)$$

Let $\varphi \in C_0^\infty(\mathbb{R}^N)$, such that $\varphi \equiv 1$ for $|\xi| \leq 1$ and $\varphi \equiv 0$ for $|\xi| > 2$. Write

$$\phi(\xi) = \varphi(\xi) - \varphi(2\xi) \quad (3.5)$$

which is supported on the spherical shell $\frac{1}{2} \leq |\xi| \leq 2$. We define

$$\delta_{\mathbf{t}}(\xi) = \prod_{i=1}^n \phi(\mathbf{t}_i \xi) \quad (3.6)$$

where the support of $\delta_{\mathbf{t}}(\xi)$ lies inside the intersection of n elliptical shells, with different homogeneities of given dilations. In particular at $\rho = 0$, it lies inside the Dyadic rectangle $|\xi_i| \sim 2^{t_i}, i = 1, 2, \dots, n$.

The partial sum operator $\Delta_{\mathbf{t}}$ is defined by

$$\begin{aligned} (\widehat{\Delta_{\mathbf{t}} f})(\xi) &= \delta_{\mathbf{t}}(\xi) \widehat{f}(\xi) \\ &= \prod_{i=1}^n \phi\left(2^{-qt_i} \xi^1, \dots, 2^{-qt_i} \xi^{i-1}, 2^{-t_i} \xi^i, 2^{-qt_i} \xi^{i+1}, \dots, 2^{-qt_i} \xi^n\right) \widehat{f}(\xi). \end{aligned} \quad (3.7)$$

For each n -tuple \mathbf{t} , we set the hypothesis below:

There exists at least one $i \in \{1, 2, \dots, n\}$ such that

$$(\mathbf{H}) \quad t_i > \frac{1}{q-1} (2 + 2 \log_2 n). \quad (3.8)$$

Without giving the explicit proof, we recall the lemma proved in [7]:

Lemma 3.1 Suppose that \mathbf{t} satisfies the hypothesis (\mathbf{H}) and $0 < \rho < 1$. Let $\mathbf{I} \cup \mathbf{J} = \{1, 2, \dots, n\}$ such that

$$(t_{i_1} + 2 + \log_2 n)/q < t_{i_2} < qt_{i_1} - (2 + \log_2 n) \quad \text{for all } i_1, i_2 \in \mathbf{I} \quad (3.9)$$

and

$$qt_j - (2 + \log_2 n) \leq t_i, \quad t_i = \max\{t_i : i \in \mathbf{I}\} \quad \text{for all } j \in \mathbf{J}. \quad (3.10)$$

Then, $\xi \in \text{supp}_{\delta_{\mathbf{t}}}(\xi)$ implies that

$$|\xi^i| \sim 2^{t_i} \quad \text{for every } i \in \mathbf{I} \quad (3.11)$$

and

$$|\xi^j| \lesssim 2^{t_j}, \quad |\xi^i| \sim 2^{t_i} \sim 2^{qt_j} \quad \text{for every } j \in \mathbf{J}. \quad (3.12)$$

Regarding to Lemma 3.1, we shall prove that

Lemma 3.2 For every given \mathbf{t} , there exists $\mathbf{I} \cup \mathbf{J} = \{1, 2, \dots, n\}$ such that (3.9)-(3.10) hold respectively.

Proof: Consider a permutation π acting on the set $\{1, 2, \dots, n\}$. We can assume that

$$t_{\pi(1)} \leq t_{\pi(2)} \leq \dots \leq t_{\pi(n)}. \quad (3.13)$$

Let $k \in \{1, 2, \dots, n\}$ such that

$$t_{\pi(k)} = \min\left\{\pi(i) : t_{\pi(n)} < qt_{\pi(i)} - (2 + \log_2 n)\right\}. \quad (3.14)$$

We thus define

$$\mathbf{I} = \{\pi(k), \pi(k+1), \dots, \pi(n)\} \quad \text{and} \quad \mathbf{J} = \{\pi(1), \pi(2), \dots, \pi(k-1)\}. \quad (3.15)$$

□

As a consequence of Lemma 3.1, we have

Lemma 3.3 *Let $\sigma \in \mathbf{S}_\rho^0$. Suppose \mathbf{t} satisfying hypothesis **(H)**, we have*

$$\left| \prod_{i=1}^n \left(\frac{\partial}{\partial \xi^i} \right)^{\alpha^i} \delta_{\mathbf{t}}(\xi) \sigma(x, \xi) \right| \leq A_\alpha \prod_{i=1}^n 2^{-t_i |\alpha^i|} \quad (3.16)$$

for every multi-index $\alpha = (\alpha^1, \alpha^2, \dots, \alpha^n)$.

Proof: Certainly, for every \mathbf{t} satisfying **(H)**, $\delta_{\mathbf{t}}(\xi)$ by definition satisfies differential inequality (3.16). Recall that $\sigma \in \mathbf{S}_\rho^0$ satisfies differential inequality (2.10) for $m = 0$. Therefore, at each i -th component, we have

$$\left| \left(\frac{\partial}{\partial \xi^i} \right)^{\alpha^i} \sigma(x, \xi) \right| \lesssim \left(\frac{1}{1 + |\xi^i| + |\xi|^\rho} \right)^{|\alpha^i|} \quad (3.17)$$

for $i = 1, 2, \dots, n$.

When $0 < \rho < 1$, by Lemma 3.1, we either have $|\xi^i| \sim 2^{t_i}$ or otherwise there exists ξ^i such that $|\xi^i| \sim 2^{t_i} \sim 2^{t_i/\rho}$, whenever $\xi \in \text{supp} \delta_{\mathbf{t}}(\xi)$. When $\rho = 0$, we have $|\xi^i| \sim 2^{t_i}$ for every $i = 1, 2, \dots, n$. \square

In order to carry out later estimation, we need a further construction on the Littlewood-Paley projections. Consider the n -tuple $\mathbf{s} = (s_1, s_2, \dots, s_n)$ whereas each $s_i, i = 1, 2, \dots, n$ is a positive integer. Let $\phi^i \in C_o^\infty(\mathbb{R}^{N_i})$ be defined as same as (3.5), on each coordinate subspace \mathbb{R}^{N_i} for $i = 1, 2, \dots, n$. We set

$$\delta_{\mathbf{t}, \mathbf{s}}(\xi) = \delta_{\mathbf{t}}(\xi) \prod_{i=1}^n \phi^i(2^{-s_i} \xi^i). \quad (3.18)$$

By Lemma 3.1, the support of $\delta_{\mathbf{t}, \mathbf{s}}(\xi)$ is either empty, or lies inside the Dyadic rectangle $\{2^{s_i-1} \leq |\xi^i| \leq 2^{s_i+1}\}$ for every $s_i \leq t_i, i = 1, 2, \dots, n$.

4 An Local L^p -Estimate for $\mathbf{N} = n$

In this section, we estimate on the L^p -boundedness of Fourier integral operator \mathcal{F} defined in (2.3)-(2.5), in the special case of $\mathbf{N}_1 = \mathbf{N}_2 = \dots = \mathbf{N}_n = 1$. Our estimation shares the same spirit of the argument given in 6.20, chapter IX of [3]. In the other hand, we need to borrow the L^p -estimate obtained in [7]. We aim to prove the following:

Theorem 4.1 *Let $\sigma \in \mathbf{S}_\rho^0$ and \mathcal{F} is the Fourier integral operator defined in (2.3)-(2.5). Suppose that $\mathbf{N} = n$. For every $f \in L^p(\mathbb{R}^{\mathbf{N}})$ has a compact support, we have*

$$\|\mathcal{F} f\|_{L^p(\mathbb{R}^{\mathbf{N}})} \lesssim \|f\|_{L^p(\mathbb{R}^{\mathbf{N}})}$$

for $1 < p < \infty$.

Proof: Recall that Φ_i is homogeneous of degree one in ξ^i , for every $i = 1, 2, \dots, n$. We shall have

$$\Phi_{\xi^i}(x^i, \xi^i) = \begin{cases} \Phi_i(x^i, +1), & \xi^i > 0 \\ \Phi_i(x^i, -1), & \xi^i < 0 \end{cases} \quad i = 1, 2, \dots, n. \quad (4.1)$$

The non-degeneracy condition (2.5) implies that

$$\frac{d\Phi_i}{dx^i}(x^i, \pm 1) \neq 0, \quad i = 1, 2, \dots, n. \quad (4.2)$$

Under appropriate smooth partition of unity, Fourier integral operator \mathcal{F} can be written as

$$\sum_{\theta} (T_{\sigma}^{\theta} f)(\Phi_1(x^1, \pm 1), \Phi_2(x^2, \pm 1), \dots, \Phi_n(x^n, \pm 1)) + (\mathcal{E}f)(x) \quad (4.3)$$

where each T_{σ}^{θ} in (4.3) is a pseudo differential operator with symbol $\sigma \in \mathbf{S}_{\rho}^0$, supported in the quadrant

$$\{\theta \cdot \xi : |\xi^i| > 1, i = 1, 2, \dots, n\} \quad (4.4)$$

where $\theta = (\text{sign}\xi^1, \text{sign}\xi^2, \dots, \text{sign}\xi^n)$. The L^p -theorem proved in [7] implies that T_{σ}^{θ} is bounded on $L^p(\mathbb{R}^N)$ for $1 < p < \infty$, among every θ in permutations.

Turn to the error term in (4.3). Let $\mathbf{I} \cup \mathbf{J}$ be any partition of the set $\{1, 2, \dots, n\}$. \mathcal{E} can be written as

$$\sum_{\mathbf{I} \cup \mathbf{J} = \{1, 2, \dots, n\}} \mathcal{E}_{\mathbf{I}, \mathbf{J}} \quad (4.5)$$

under some suitable smooth partition of unity, whereas the symbol of each $\mathcal{E}_{\mathbf{I}, \mathbf{J}}$ is supported in the quadrant

$$\{\theta \cdot \xi : |\xi^i| > 1, i \in \mathbf{I} \text{ and } |\xi^j| \lesssim 1, j \in \mathbf{J}\} \quad (4.6)$$

for which $|\mathbf{J}|$ is at least 1. By taking local diffeomorphisms $x^i \rightarrow \Phi_i(x^i, \pm 1)$ for every $i \in \mathbf{I}$, each $\mathcal{E}_{\mathbf{I}, \mathbf{J}}$ can be written as

$$\begin{aligned} (\mathcal{E}_{\mathbf{I}, \mathbf{J}} f)(x) &= \int e^{2\pi i x \cdot \xi} \sigma_{\mathbf{I}, \mathbf{J}}(x, \xi) \widehat{f}(\xi) d\xi \\ &= \int e^{2\pi i x \cdot \xi} \left\{ \int \prod_{j \in \mathbf{J}} e^{2\pi i (\Phi_j(x^j, \xi^j) - x^j \xi^j)} \varphi_{\mathbf{I}, \mathbf{J}}(\xi) \sigma(x, \xi) \right\} \widehat{f}(\xi) d\xi \end{aligned} \quad (4.7)$$

where $\varphi_{\mathbf{I}, \mathbf{J}}(\xi) \in C^{\infty}(\mathbb{R}^N)$ equals 1 in $\bigoplus_{i \in \mathbf{I}} \mathbb{R}^{N_i}$ and has compact support in $\bigoplus_{j \in \mathbf{J}} \mathbb{R}^{N_j}$.

Observe that $\mathcal{E}_{\mathbf{I}, \mathbf{J}}$ is a pseudo differential operator with symbol $\sigma_{\mathbf{I}, \mathbf{J}}(x, \xi)$, showed implicitly in (4.7). The size of $\sigma_{\mathbf{I}, \mathbf{J}}$ is uniformly bounded provided that $\sigma \in \mathbf{S}_{\rho}^0$. Moreover, it satisfies differential inequality (2.10) on the subspace $\bigoplus_{i \in \mathbf{I}} \mathbb{R}^{N_i} \times \mathbb{R}^{N_i}$. In the other hand, $\sigma_{\mathbf{I}, \mathbf{J}}$ has a compact support in the subspace $\bigoplus_{j \in \mathbf{J}} \mathbb{R}^{N_j} \times \mathbb{R}^{N_j}$, since σ has a x -compact support and $\varphi_{\mathbf{I}, \mathbf{J}}$ has a ξ -compact support in $\bigoplus_{j \in \mathbf{J}} \mathbb{R}^{N_j}$. The corresponding kernel on the subspace belongs to $C_o^{\infty}(\bigoplus_{j \in \mathbf{J}} \mathbb{R}^{N_j})$. By carrying out the same estimation given in section 5 of [7], we shall have each $\mathcal{E}_{\mathbf{I}, \mathbf{J}}$ bounded on $L^p(\mathbb{R}^N)$ for $1 < p < \infty$. \square

5 L^2 -Estimate

By Plancherel's theorem, our estimate is reduced to a similar assertion for the operator

$$(Sf)(x) = \int e^{2\pi i \Phi(x, \xi)} \sigma(x, \xi) f(\xi) d\xi. \quad (5.1)$$

Let c be a small, positive constant. We define the *narrow cone* inside the frequency space in the following way: whenever ξ and η belong to the cone and $|\eta| \leq |\xi|$, the component of η which is perpendicular to ξ , denoted by η^\perp , satisfies $|\eta^\perp| \leq c|\xi|$. It is clear that any given S can be written as a finite sum of operators where $\sigma(x, \xi)$ is supported on the narrow cone, by using a suitable smooth partition of unity.

We claim that

$$|\nabla_x (\Phi(x, \xi) - \Phi(x, \eta))| \geq \text{const } |\xi - \eta| \quad (5.2)$$

whenever ξ and η belong to the same narrow cone. For the result in (5.2), we refer to the estimates given in section 3, chapter IX of [3].

The operator S defined in Sf has a dual operator

$$(S^*f)(\xi) = \int e^{-2\pi i \Phi(x, \xi)} \overline{\sigma}(x, \xi) f(x) dx. \quad (5.3)$$

Let Ω^\sharp denote the kernel of operator S^*S . A direct computation shows that

$$(S^*Sf)(\xi) = \int f(\eta) \Omega^\sharp(\xi, \eta) d\eta \quad (5.4)$$

where

$$\Omega^\sharp(\xi, \eta) = \int e^{2\pi i (\Phi(x, \eta) - \Phi(x, \xi))} \sigma(x, \eta) \overline{\sigma}(x, \xi) dx. \quad (5.5)$$

Since $\sigma(x, \xi)$ has x -compact support, Ω^\sharp is bounded. Recall estimate (5.2) and $\Phi(x, \xi)$ is homogeneous of degree one in ξ . Integration by parts with respect to x in (5.5) gives

$$|\Omega^\sharp(\xi, \eta)| \lesssim |\xi - \eta|^{-2N} \left| \int e^{2\pi i (\Phi(x, \eta) - \Phi(x, \xi))} \Delta_x^N (\sigma(x, \eta) \overline{\sigma}(x, \xi)) dx \right| \quad (5.6)$$

for $\xi \neq \eta$ and every $N \geq 1$. By differential inequality (2.10), estimate (5.6) implies that

$$|\Omega^\sharp(\xi, \eta)| \leq A_N (1 + |\xi - \eta|)^{-2(1-\rho)N} \quad (5.7)$$

for every $N \geq 1$.

Estimate (5.7) indeed implies that S^*S is bounded on L^2 -spaces, and therefore S is provided that N is sufficiently large depending on $0 \leq \rho < 1$. An regarding estimate can be found in the lemma given at 2.4.1, chapter VII of [3].

As a consequence of L^2 -boundedness of Fourier integral operator \mathcal{F} , we can obtain the following variant of fractional integration:

Lemma 5.1 Let \mathcal{F} be defined as (2. 3)-(2. 5) whose symbol $\sigma \in \mathbf{S}_\rho^m$ with $-\mathbf{N}/2 < m < 0$. Then, \mathcal{F} initially defined on \mathcal{S} , extends to a bounded operator from $\mathbf{L}^p(\mathbb{R}^{\mathbf{N}})$ to $\mathbf{L}^2(\mathbb{R}^{\mathbf{N}})$ whenever

$$\frac{1}{p} = \frac{1}{2} - \frac{m}{\mathbf{N}}, \quad (5. 8)$$

and extends to a bounded operator from $\mathbf{L}^2(\mathbb{R}^{\mathbf{N}})$ to $\mathbf{L}^q(\mathbb{R}^{\mathbf{N}})$ whenever

$$\frac{1}{q} = \frac{1}{2} + \frac{m}{\mathbf{N}}. \quad (5. 9)$$

Proof: The $\mathbf{L}^p \longrightarrow \mathbf{L}^2$ boundedness of \mathcal{F} follows that

$$(\mathcal{F}f)(x) = \int e^{2\pi i \Phi(x, \xi)} \sigma(x, \xi) (1 + |\xi|^2)^{-\frac{m}{2}} \widehat{Tf}(\xi) d\xi \quad (5. 10)$$

where $\widehat{Tf}(\xi) = (1 + |\xi|^2)^{\frac{m}{2}} \widehat{f}(\xi)$ is a singular integral of convolution type. Its kernel satisfies a size estimate of

$$|\Omega(x)| \lesssim |x|^{-\mathbf{N}-m}. \quad (5. 11)$$

By Hardy-Littlewood-Sobolev inequality, we have $\|\mathcal{F}a\|_{\mathbf{L}^2} \lesssim \|a\|_{\mathbf{L}^p}$ if and only if

$$\frac{1}{p} = \frac{1}{2} - \frac{m}{\mathbf{N}}.$$

Turn to the other hand. Let us write $\mathcal{F} = ST$ and note that $\mathcal{F}^* = T^*S^*$. It is then enough to see that $S^* : \mathbf{L}^p \longrightarrow \mathbf{L}^2$ where p and q are conjugate exponents. Since $\|S^*f\|_{\mathbf{L}^2}^2 = \langle SS^*f, f \rangle$. By Hölder inequality, it is suffice to prove that $SS^* : \mathbf{L}^p \longrightarrow \mathbf{L}^q$. A direct computation shows that

$$(SS^*f)(x) = \int f(y) \Omega^b(x, y) dy \quad (5. 12)$$

where

$$\begin{aligned} \Omega^b(x, y) &= \int e^{2\pi i (\Phi(x, \xi) - \Phi(y, \xi))} \sigma(x, \xi) \overline{\sigma}(y, \xi) d\xi \\ &= \int \dots \int \prod_{i=1}^n e^{2\pi i (\Phi_i(x^i, \xi^i) - \Phi_i(y^i, \xi^i))} \sigma(x, \xi) \overline{\sigma}(y, \xi) d\xi^1 \dots \xi^n. \end{aligned} \quad (5. 13)$$

Recall that $\mathbf{N} = \mathbf{N}_1 + \mathbf{N}_2 + \dots + \mathbf{N}_n$. Let $m = m_1 + m_2 + \dots + m_n$ such that $m_i/\mathbf{N}_i = m/\mathbf{N}$ for every $i = 1, 2, \dots, n$. By carrying out the estimation in analogue to 3.1.4, chapter IX of [3], inductively on each subspace $\mathbb{R}^{\mathbf{N}_i}$, $i = 1, 2, \dots, n$, we have

$$|\Omega^b(x, y)| \lesssim \prod_{i=1}^n |x^i - y^i|^{-\mathbf{N}_i - 2m_i}. \quad (5. 14)$$

By applying Hardy-Littlewood-Sobolev inequality on each subspace $\mathbb{R}^{\mathbf{N}_i}$, $i = 1, 2, \dots, n$, and using Minkowski inequality, we conclude that $SS^* : \mathbf{L}^p \longrightarrow \mathbf{L}^q$ whenever

$$\frac{1}{q} = \frac{1}{p} + \frac{2m}{\mathbf{N}}. \quad (5. 15)$$

By taking into account that $1/p + 1/q = 1$, we obtain (5. 9) as a desired result. \square

6 A Heuristic Argument

In order to obtain our desired \mathbf{L}^p -estimate, the main point is to show that the operator \mathcal{F} defined in (2. 3)-(2. 5) is bounded from \mathbf{H}^1 to \mathbf{L}^1 , provide that $\sigma \in \mathbf{S}_\rho^m$ for $m = -(\mathbf{N} - n)/2$. Let $\mathbf{B}_\delta \subset \mathbb{R}^{\mathbf{N}}$ be the ball centered on $x_0 = (x_0^1, x_0^2, \dots, x_0^n)$ with radius $\delta > 0$. We consider a as an \mathbf{H}^1 -atom in $\mathbb{R}^{\mathbf{N}}$, such that

$$\left\{ \begin{array}{l} \text{supp } a \subset \mathbf{B}_\delta, \\ |a(x)| \leq |\mathbf{B}_\delta|^{-1} \quad \text{for almost every } x, \\ \int a(x) dx = 0. \end{array} \right. \quad (6. 1)$$

As was classified in [11], $f \in \mathbf{H}^1$ can be written as $\sum \lambda_k a_k$, whereas each a_k is an \mathbf{H}^1 -atom associated to the ball \mathbf{B}_{δ_k} and $\sum |\lambda_k| < \mathbf{const}$. Therefore, it is suffice to show that

$$\int |\mathcal{F}a(x)| dx \leq \mathbf{const}. \quad (6. 2)$$

Let $\delta > 0$. The pseudo-local property of \mathcal{F} eliminated us to consider so-called the *region of influence* with respect to \mathbf{B}_δ , denoted by \mathbf{B}_δ^* , which is essentially the set $\{x : \mathbf{dist}(\Sigma_x, \mathbf{B}_\delta) \lesssim \delta\}$. Since Σ_x defined in (2. 9) could be very singular, and has dimension at most equal to $\mathbf{N} - n$, we shall take

$$|\mathbf{B}_\delta^*| \lesssim \delta^n \quad (6. 3)$$

for δ is small. By Schwartz inequality, we have

$$\int_{\mathbf{B}_\delta^*} |\mathcal{F}a(x)| dx \leq |\mathbf{B}_\delta^*|^{\frac{1}{2}} \|\mathcal{F}a\|_{\mathbf{L}^2} \lesssim \delta^{\frac{n}{2}} \|\mathcal{F}a\|_{\mathbf{L}^2}. \quad (6. 4)$$

From estimate (5. 8) in Lemma 5.1, we have $\|\mathcal{F}a\|_{\mathbf{L}^2} \lesssim \|a\|_{\mathbf{L}^p}$ whenever

$$\frac{1}{p} = \frac{1}{2} + \frac{\mathbf{N} - n}{2\mathbf{N}}. \quad (6. 5)$$

The \mathbf{H}^1 -atom a defined in (6. 1) satisfies $\|a\|_{\mathbf{L}^p} \leq |\mathbf{B}|^{-1+\frac{1}{p}}$ for $1 < p < \infty$. Together with estimate in (6. 5), we have

$$\int_{\mathbf{B}_\delta^*} |\mathcal{F}a(x)| dx \lesssim \delta^{\frac{n}{2}} \delta^{\mathbf{N}(-1+\frac{1}{p})} = 1. \quad (6. 6)$$

The implied constant depends on $0 \leq \rho < 1$.

In the other hand, let \mathcal{F}^* be the dual operator of \mathcal{F} . By estimate (5. 9) in Lemma 5.1, it follows by duality that \mathcal{F}^* satisfies the above estimates. We have

$$\int_{\mathbf{B}_\delta^*} |\mathcal{F}^*a(x)| dx \leq \mathbf{const}. \quad (6. 7)$$

Notice that the exact set of \mathbf{B}_δ^* will be constructed explicitly for \mathcal{F} and \mathcal{F}^* respectively.

Once we show that estimates (6. 6)-(6. 7) hold for the complement of \mathbf{B}_δ^* , the \mathbf{L}^p -estimate will be concluded by applying the complex interpolation theorem set out in 5.2, chapter IV of [3]. Consider the analytic family of operators \mathcal{F}_s in the strip $0 < \mathbf{Re}(s) \leq 1$, by

$$\mathcal{F}_s(x) = e^{(s-\vartheta)^2} \int e^{2\pi i \Phi(x, \xi)} \sigma(x, \xi) (1 + |\xi|^2)^{\frac{\gamma(s)}{2}} \widehat{f}(\xi) d\xi \quad (6. 8)$$

where

$$\gamma(s) = -m - \frac{s(\mathbf{N} - n)}{2}, \quad \vartheta = -\frac{2m}{\mathbf{N} - n}. \quad (6. 9)$$

Let $s = u + it$. There are only finitely many derivatives of the symbol involved in our previous estimation. Therefore, the corresponding derivatives have at most a polynomial growth in t , whereas the factor $e^{(s-\vartheta)^2}$ decays rapidly as $|t| \rightarrow \infty$.

When $\mathbf{Re}(s) = 0$, $\sigma(x, \xi) (1 + |\xi|^2)^{\frac{\gamma(s)}{2}}$ has an order of zero. As a result of the \mathbf{L}^2 -estimate in section 5, we have

$$\|\mathcal{F}_{it}f\|_{\mathbf{L}^2} \leq A_\rho \|f\|_{\mathbf{L}^2}, \quad -\infty < t < \infty. \quad (6. 10)$$

When $\mathbf{Re}(s) = 1$, $\sigma(x, \xi) (1 + |\xi|^2)^{\frac{\gamma(s)}{2}}$ has an order of $-(\mathbf{N} - n)/2$. If $\mathcal{F}: \mathbf{H}^1 \rightarrow \mathbf{L}^1$, as was first classified in [12], the duality between \mathbf{H}^1 and \mathbf{BMO} implies that

$$\|\mathcal{F}_{1+it}f\|_{\mathbf{BMO}} \lesssim \|f\|_{\mathbf{L}^\infty}, \quad -\infty < t < \infty, \quad f \in \mathbf{L}^2 \cap \mathbf{L}^\infty. \quad (6. 11)$$

By the desired complex interpolation, we obtain

$$\|\mathcal{F}_\vartheta f\|_{\mathbf{L}^p} \leq A_{\vartheta, \rho} \|f\|_{\mathbf{L}^p} \quad (6. 12)$$

where $\vartheta = 1 - 2/p > 0$. Observe that

$$\mathcal{F}_\vartheta = \mathcal{F} \quad (6. 13)$$

for which $1/2 - 1/p = -m/(\mathbf{N} - n)$.

In the other hand, \mathcal{F}^* is bounded on \mathbf{L}^2 since \mathcal{F} is, as proved in section 5. If $\mathcal{F}^*: \mathbf{H}^1 \rightarrow \mathbf{L}^2$, the above estimates in (6. 11)-(6. 13) shall be also valid for \mathcal{F}^* . By duality, \mathcal{F} is bounded on $\mathbf{L}^p(\mathbb{R}^N)$, whenever

$$\left| \frac{1}{2} - \frac{1}{p} \right| \leq \frac{-m}{\mathbf{N} - n}. \quad (6. 14)$$

What remains to be shown is that \mathcal{F} and \mathcal{F}^* are bounded from \mathbf{H}^1 to \mathbf{L}^1 , on the complement of their region of influences.

Construction of \mathbf{B}_δ^* :

Set

$$\mathbf{U} \cup \mathbf{V} = \{1, 2, \dots, n\} \quad (6. 15)$$

$$\mathbf{N}_i > 1 \text{ for } i \in \mathbf{U}, \quad \mathbf{N}_j = 1 \text{ for } j \in \mathbf{V}.$$

From now on, we will assume that there is at least one $i \in \{1, 2, \dots, n\}$ such that $\mathbf{N}_i \geq 2$. Let $\mathbb{S}^{\mathbf{N}_i-1}$ be the unit sphere in the coordinate subspace $\mathbb{R}^{\mathbf{N}_i}$, for $i \in \mathbf{U}$. We consider an equally distributed set of points on $\mathbb{S}^{\mathbf{N}_i-1}$ with grid length equal to $2^{-s_i/2}$. Denote the collection of

such points by $\{\xi_{s_i}^{v_i}\}_{v_i}$. It is clear that there are at most a constant multiple of $2^{(N_i-1)/2}$ elements in $\{\xi_{s_i}^{v_i}\}_{v_i}$, for every $s_i \geq 1$.

We first construct \mathbf{B}_δ^* with respect to \mathcal{F} . Define the *rectangles* by

$$R_{s_i}^{v_i} \doteq \left\{ x^i : \left| x_0^i - \Phi_{\xi^i}(x^i, \xi_{s_i}^{v_i}) \right| \lesssim 2^{-s_i/2}, \left| \pi_{s_i}^{v_i}(x_0^i - \Phi_{\xi^i}(x^i, \xi_{s_i}^{v_i})) \right| \lesssim 2^{-s_i} \right\} \quad (6.16)$$

for every $i \in \mathbf{U}$, where $\pi_{s_i}^{v_i}$ is the orthogonal projection in the direction of $\xi_{s_i}^{v_i}$. The region of influence \mathbf{B}_δ^* is defined by

$$\mathbf{B}_\delta^* = \bigcup_{i \in \mathbf{U}} \left\{ \bigcup_{2^{-s_i} \leq \delta} \bigcup_{v_i} R_{s_i}^{v_i} \right\}. \quad (6.17)$$

The set \mathbf{B}_δ^* in (6.17) satisfies the norm estimate (6.3) for $\delta < 1$, provided that $|\mathbf{U}| \leq n$. For regarding details, please see 4.3, chapter IX of [3].

We construct \mathbf{B}_δ^* with respect to \mathcal{F}^* by replacing $R_{s_i}^{v_i}$ in (6.16) with

$$^*R_{s_i}^{v_i} \doteq \left\{ x^i : \left| x_0^i - \Phi_{\xi^i}(x_0^i, \xi_{s_i}^{v_i}) \right| \lesssim 2^{-s_i/2}, \left| \pi_{s_i}^{v_i}(x_0^i - \Phi_{\xi^i}(x_0^i, \xi_{s_i}^{v_i})) \right| \lesssim 2^{-s_i} \right\} \quad (6.18)$$

for every $i \in \mathbf{U}$, where $\pi_{s_i}^{v_i}$ is the orthogonal projection in the direction of $\xi_{s_i}^{v_i}$. The *region of influence* \mathbf{B}_δ^* will be defined as same as (6.17), but in terms of $^*R_{s_i}^{v_i}$ in (6.18).

Construction of partition of unity

Let $\Gamma_{s_i}^{v_i}$ denotes the cone whose central direction is $\xi_{s_i}^{v_i}$, such that

$$\Gamma_{s_i}^{v_i} = \left\{ \xi^i : \left| \frac{\xi^i}{|\xi^i|} - \xi_{s_i}^{v_i} \right| \leq 2 \times 2^{-s_i/2} \right\} \quad (6.19)$$

for every $i \in \mathbf{U}$. Recall from 4.4, chapter IX of [3]. We can construct an associated partition of unity $\chi_{s_i}^{v_i}$, with homogeneity zero in ξ^i and supported in $\Gamma_{s_i}^{v_i}$. Moreover,

$$\sum_{v_i} \chi_{s_i}^{v_i}(\xi^i) = 1 \quad \text{for } \xi^i \neq 0 \text{ and all } s_i \geq 1 \quad (6.20)$$

$$\left| \left(\frac{\partial}{\partial \xi^i} \right)^{\alpha^i} \chi_{s_i}^{v_i}(\xi^i) \right| \leq A_{\alpha^i} 2^{|\alpha^i|s_i/2} |\xi^i|^{-|\alpha^i|}.$$

Lastly, we define simultaneously

$$\chi_s^v(\xi) = \prod_{i \in \mathbf{I}} \chi_{s_i}^{v_i}(\xi^i) (1 - \psi(\xi)) \quad (6.21)$$

and

$$^c \chi_s^v(\xi) = \prod_{i \in \mathbf{I}} \chi_{s_i}^{v_i}(\xi^i) \psi(\xi) \quad (6.22)$$

where $\psi \in C^\infty(\mathbb{R}^N)$ equals 1 on the subspaces $\xi^i = 0$, $i = 1, 2, \dots, n$ and vanishes on the quadrants $\{|\xi^i| > 1; i = 1, 2, \dots, n\}$. In other words, ψ in (6.21)-(6.22) is supported within a bounded distance from the coordinate axes.

7 L^p -Estimate

The L^p -estimate for $N > n$ will be carried out in analogue to the estimation in chapter IX of [3]. Recall from the previous section. We need to show that

$$\int_{\epsilon \mathbf{B}_\delta^*} |\mathcal{F}a(x)| dx \leq \mathbf{const} \quad (7.1)$$

whenever a is an \mathbf{H}^1 -atom. A direct computation shows that

$$(\mathcal{F} \Delta_{\mathbf{t}} f)(x) = \int f(y) \Omega_{\mathbf{t}}(x, y) dy \quad (7.2)$$

where

$$\Omega_{\mathbf{t}}(x, y) = \int e^{2\pi i(\Phi(x, \xi) - y \cdot \xi)} \delta_{\mathbf{t}}(\xi) \sigma(x, \xi) d\xi. \quad (7.3)$$

Observe that by definition of $\delta_{\mathbf{t}}(\xi)$ in (3.6), we have

$$\Omega = \sum_{\mathbf{t}} \Omega_{\mathbf{t}} + \Omega_o$$

where \mathbf{t} in the summation satisfies hypothesis **(H)** in (3.8). It is easy to observe that $\Omega_o \in C^\infty(\mathbb{R}^N \times \mathbb{R}^N)$ and decays rapidly at infinity.

Let j be a positive integer such that $2^{j-1} \leq |\xi| \leq 2^{j+1}$ for $\xi \in \mathbf{supp} \delta_{\mathbf{t}}(\xi)$. By Lemma 3.1, we must have $t_i \leq j$ for every $i = 1, 2, \dots, n$. In particular, we write

$$\Omega_{\mathbf{t}} = \Omega_j \quad (7.4)$$

for $t_1 = t_2 = \dots = t_n = j$.

The main objective in this section is proving the following estimates of the kernel $\Omega_{\mathbf{t}}$:

Lemma 7.1 Suppose $\sigma \in \mathbf{S}_\rho^m$ with $m = -(N - n)/2$. We have

$$\int |\Omega_{\mathbf{t}}(x, y)| dx \lesssim \prod_{i=1}^n 2^{(t_i - j)(N_i - 1)/2} \quad \text{for } y \in \mathbb{R}^N, \quad (7.5)$$

$$\int |\Omega_j(x, y) - \Omega_j(x, z)| dx \lesssim 2^j |y - z| \quad \text{for } y, z \in \mathbb{R}^N \quad (7.6)$$

and

$$\int_{\epsilon \mathbf{B}_\delta^*} |\Omega_j(x, y)| dx \lesssim \frac{2^{-j}}{\delta} \quad \text{for } y \in \mathbf{B}_\delta \quad (7.7)$$

whenever $2^j > \delta^{-1}$.

Remark 7.1 By assumption, there is at least one $i \in \{1, 2, \dots, n\}$ such that $N_i \geq 2$. We thus have $m = -(N - n)/2 \leq -1/2$. In fact, suppose $\Phi(x, \xi) = x \cdot \xi$. Then, \mathcal{F} defined in (2.3) with its symbol $\sigma \in \mathbf{S}_\rho^{-\varepsilon}$ for any $\varepsilon > 0$, satisfies the weak type-(1,1) estimate.

Proof : First, for each \mathbf{t} satisfying **(H)** in (3. 8), we write

$$\Omega_{\mathbf{t}} = \sum_{\mathbf{s}} \Omega_{\mathbf{t},\mathbf{s}} + {}^c\Omega_{\mathbf{t},\mathbf{s}} \quad (7. 8)$$

where

$$\begin{aligned} \Omega_{\mathbf{t},\mathbf{s}}(x, y) &= \sum_{\nu} \Omega_{\mathbf{t},\mathbf{s}}^{\nu}(x, y) \\ &= \sum_{\nu} \int e^{2\pi i(\Phi(x,\xi) - y \cdot \xi)} \chi_{\mathbf{s}}^{\nu}(\xi) \delta_{\mathbf{t},\mathbf{s}}(\xi) \sigma(x, \xi) d\xi \end{aligned} \quad (7. 9)$$

and

$$\begin{aligned} {}^c\Omega_{\mathbf{t},\mathbf{s}}^{\nu}(x, y) &= \sum_{\nu} {}^c\Omega_{\mathbf{t},\mathbf{s}}^{\nu}(x, y) \\ &= \sum_{\nu} \int e^{2\pi i(\Phi(x,\xi) - y \cdot \xi)^c} \chi_{\mathbf{s}}^{\nu}(\xi) \delta_{\mathbf{t},\mathbf{s}}(\xi) \sigma(x, \xi) d\xi. \end{aligned} \quad (7. 10)$$

Notice that $\delta_{\mathbf{t},\mathbf{s}}(\xi)$ is defined in (3. 18) and $\chi_{\mathbf{s}}^{\nu}$, ${}^c\chi_{\mathbf{s}}^{\nu}$ are defined respectively in (6. 21)-(6. 22).

Let $\mathbf{U} \cup \mathbf{V} = \{1, 2, \dots, n\}$ be defined as (6. 15). In each i -th subspace for $i \in \mathbf{U}$, we can choose a new framework under some appropriate linear transformation, such that the first coordinate coincides with the direction of $\xi_{s_i}^{\nu_i}$, whenever $\xi \in \text{supp} \chi_{\mathbf{s}}^{\nu}(\xi) \delta_{\mathbf{t},\mathbf{s}}(\xi)$. It should be clear that differential inequality (2. 10) remains to be true.

With a bit omitted on notations, we then write

$$\Phi_i(x^i, \xi^i) - y^i \cdot \xi^i = (\Phi_{\xi^i}(x^i, \xi_{*}^i) - y^i) \cdot \xi^i + (\Phi_i(x^i, \xi^i) - \Phi_{\xi^i}(x, \xi_{*}^i) \cdot \xi^i) \quad (7. 11)$$

where $\xi_{*}^i = \xi_1^i / |\xi_1^i|$ is in the direction of $\xi_{s_i}^{\nu_i}$, for every $i \in \mathbf{U}$. Recall Φ_{ξ^i} is homogeneous of degree zero in ξ^i . Define

$$\varphi_i(x^i, \xi^i) = \Phi_i(x^i, \xi^i) - \Phi_{\xi^i}(x, \xi_{*}^i) \cdot \xi^i, \quad i \in \mathbf{U}. \quad (7. 12)$$

Let $\xi^i = (\tau^i, \eta^i) \in \mathbb{R} \times \mathbb{R}^{\mathbf{N}_i-1}$ for every $i \in \mathbf{U}$ such that

$$\tau^i = \xi_1^i \in \mathbb{R}, \quad \eta^i = (\xi_1^i)^{\dagger} \in \mathbb{R}^{\mathbf{N}_i-1}. \quad (7. 13)$$

By carrying out the estimation given in 4.5, chapter IX of [3], we have

$$\left| \left(\frac{\partial}{\partial \tau^i} \right)^N \varphi_i(x^i, \xi^i) \right| \leq A_N 2^{-N s_i}, \quad \left| (\nabla_{\eta^i})^N \varphi_i(x^i, \xi^i) \right| \leq A_N 2^{-N s_i/2} \quad (7. 14)$$

for every $N \geq 1$.

Let

$$\omega_{\mathbf{t},\mathbf{s}}^{\nu}(x, \xi) = \prod_{i \in \mathbf{U}} e^{2\pi i \varphi_i(x^i, \xi^i)} \chi_{\mathbf{s}}^{\nu}(\xi) \delta_{\mathbf{t},\mathbf{s}}(\xi) \sigma(x, \xi). \quad (7. 15)$$

We rewrite $\Omega_{\mathbf{t},\mathbf{s}}^{\nu}$ defined implicitly in (7. 9) as

$$\int \left(\prod_{j \in \mathbf{V}} e^{2\pi i(\Phi_j(x^j, \xi^j) - y^j \cdot \xi^j)} \right) \left(\prod_{i \in \mathbf{U}} e^{2\pi i(\Phi_{\xi^i}(x^i, \xi_{*}^i) - y^i \cdot \xi^i)} \right) \omega_{\mathbf{t},\mathbf{s}}^{\nu}(x, \xi) d\xi. \quad (7. 16)$$

Let $\sigma \in \mathbf{S}_p^m$ and has an order of $m = -(\mathbf{N} - n)/2$. If $|\xi| \sim 2^j$, we have

$$|\omega_{\mathbf{t},s}^v(x, \xi)| \lesssim \left(\frac{1}{1 + |\xi|} \right)^{\frac{\mathbf{N}-n}{2}} \sim 2^{-j(\mathbf{N}-n)/2}. \quad (7.17)$$

The rest of estimation will be accomplished in several steps.

1. Define the differential operators

$$\begin{cases} L^i = I - 2^{2s_i} \left(\frac{\partial}{\partial \tau^i} \right)^2 - 2^{s_i} \Delta_{\eta^i}, & i \in \mathbf{U}, \\ L^j = I - 2^{2s_j} \Delta_{\xi^j}, & j \in \mathbf{V}. \end{cases} \quad (7.18)$$

Let $\xi \in \text{supp } \chi_s^v(\xi) \delta_{\mathbf{t},s}(\xi)$. Recall estimates (7.14)-(7.17) and (6.20), together with Lemma 3.3. We have

$$\left| (L^j)^M (L^i)^N (\omega_{\mathbf{t},s}^v(x, \xi)) \right| \leq A_{M,N} 2^{-j(\mathbf{N}-n)/2} \quad (7.19)$$

for every $i \in \mathbf{U}$, $j \in \mathbf{V}$ and every $M \geq 1$, $N \geq 1$.

Turning to the other side,

$$\begin{aligned} (L^i)^N \left(e^{2\pi i (\Phi_{\xi^i}(x^i, \xi_*^i) - y^i) \cdot \xi^i} \right) = \\ \left\{ 1 + 4\pi^2 2^{2s_i} \left| (\Phi_{\xi^i}(x^i, \xi_*^i) - y^i)_1 \right|^2 + 4\pi^2 2^{s_i} \left| (\Phi_{\xi^i}(x^i, \xi_*^i) - y^i)_1^\dagger \right|^2 \right\}^N e^{2\pi i (\Phi_{\xi^i}(x^i, \xi_*^i) - y^i) \cdot \xi^i} \end{aligned} \quad (7.20)$$

for every $i \in \mathbf{U}$. In the support of χ_s^v defined in (6.21), the non-degeneracy condition (2.5) implies the local diffeomorphisms

$$x^i \longrightarrow \Phi_{\xi^i}(x^i, \xi_*^i), \quad i \in \mathbf{U} \quad (7.21)$$

whose Jacobians are bounded from below.

In the other hand, we have

$$\begin{aligned} (L^j)^M \left(e^{2\pi i (\Phi_j(x^j, \xi^j) - y^j) \cdot \xi^j} \right) = \\ \left\{ 1 + 4\pi^2 2^{2s_j} \left| (\Phi_j(x^j, \pm 1) - y^j) \right|^2 \right\}^M e^{2\pi i (\Phi_j(x^j, \xi^j) - y^j) \cdot \xi^j} \end{aligned} \quad (7.22)$$

for every $j \in \mathbf{V}$.

By definition in (6.21), $\chi_s^v(\xi)$ is supported in a distance away from the coordinate subspaces $\xi^i = 0$, $i = 1, 2, \dots, n$. The non-degeneracy condition (2.5) implies the local diffeomorphisms

$$x^j \longrightarrow \Phi_j(x^j, \pm 1), \quad j \in \mathbf{V} \quad (7.23)$$

as in Section 4.

By definitions in (6. 19)-(6. 21), the support of $\chi_s^v(\xi)\delta_{t,s}(\xi)$ has a size bounded by

$$\prod_{i \in \mathbf{U}} 2^{s_i} 2^{s_i(\mathbf{N}_i-1)/2} \times \prod_{j \in \mathbf{V}} 2^{s_j} \quad (7. 24)$$

with $s_i \leq t_i$ for every $i = 1, 2, \dots, n$.

By putting all together the estimates (7. 19)-(7. 24), with N and M sufficiently large, we have

$$\begin{aligned} & \int |\Omega_{t,s}^v(x, y)| dx \\ & \lesssim \prod_{i \in \mathbf{U}} 2^{s_i} 2^{(s_i-j)(\mathbf{N}_i-1)/2} \int_{\mathbb{R}^{\mathbf{N}_i}} \left(1 + 2^{s_i} \left| (x^i - y^i)_1 \right| + 2^{s_i/2} \left| (x^i - y^i)_1^\dagger \right| \right)^{-2N} dx^i \\ & \quad \times \prod_{j \in \mathbf{V}} 2^{\mathbf{N}_j s_j} \int_{\mathbb{R}^{\mathbf{N}_j}} \left(1 + 2^{s_j} \left| (x^j - y^j) \right| \right)^{-2M} dx^j \\ & \lesssim 2^{-J(\mathbf{N}-n)/2}. \end{aligned} \quad (7. 25)$$

Next, we claim that ${}^c\Omega_{t,s}^v$ in (7. 10) satisfies the same estimate as above. By definition of ${}^c\chi_s^v$ in (6. 22), the integrant of (7. 10) has compact support in at least one coordinate subspace. On its complement, we can carry out the exactly same estimation as before. Since $\sigma(x, \xi)$ has compact support in x , integration in the extra dimensions within the compact support will have no impact on the size of ${}^c\Omega_{t,s}^v$, other than a multiple of constant. Therefore, the estimate in (7. 25) is also valid for ${}^c\Omega_{t,s}^v$.

Recall from the construction of \mathbf{B}_δ^* in the previous section, there are at most a constant multiple of

$$\prod_{i \in \mathbf{U}} 2^{s_i(\mathbf{N}_i-1)/2} \quad (7. 26)$$

elements in the collection of v . We have

$$\begin{aligned} \int |\Omega_t(x, y)| dx & \lesssim \sum_{\mathbf{s}} \sum_v \int |\Omega_{t,s}^v(x, y)| + |{}^c\Omega_{t,s}^v(x, y)| dx \\ & \lesssim \sum_{s_i \leq t_i \leq j} \left\{ \prod_{i=1}^n 2^{(s_i-j)(\mathbf{N}_i-1)/2} \right\} \\ & \lesssim \prod_{i=1}^n 2^{(t_i-j)(\mathbf{N}_i-1)/2} \end{aligned} \quad (7. 27)$$

uniformly in y .

2. Suppose $t_1 = t_2 = \dots = t_n = j$. We have $|\xi^i| \sim 2^j$ for every $i = 1, 2, \dots, n$, whenever $\xi \in \delta_t(\xi)$ by Lemma 4.1. In this case, we can assume $\mathbf{t} = \mathbf{s}$ since the support of $\delta_{t,s}(\xi)$ by definition is nonempty only if $|\mathbf{t} - \mathbf{s}| = \sum |t_i - s_i| < \mathbf{const}$.

Observe that a differentiation with respect to y in (7. 16) gives a factor bounded by 2^j . By carrying out the same estimation in step 1, we obtain

$$\int |\nabla_y \Omega_j(x, y)| dx \lesssim 2^j. \quad (7. 28)$$

Therefore, for every $y, z \in \mathbb{R}^N$, we have

$$\int |\Omega_j(x, y) - \Omega_j(x, z)| dx \lesssim 2^j |y - z|. \quad (7. 29)$$

3. Suppose $t_1 = t_2 = \dots = t_n = j$. Let k be a positive integer such that $2^{k-1} \leq \delta \leq 2^{k+1}$. For $x \in {}^c\mathbf{B}_\delta^*$, by definition in (6. 16)-(6. 17), there exists at least one x^i for some $i \in \mathbf{U}$, such that

$$\left| \pi_j^{v_i} (\Phi_{\xi^i}(x^i, \xi_j^i) - x_o^i) \right| \gtrsim 2^{-k} \quad \text{and} \quad \left| \Phi_{\xi^i}(x^i, \xi_j^i) - x_o^i \right| \gtrsim 2^{-k/2}. \quad (7. 30)$$

If $y \in \mathbf{B}_\delta$, then $|y - x_o| \leq 2^{-k+1}$. Under appropriate linear transformations as above, we have

$$2^j \left| (\Phi_{\xi^i}(x^i, \xi_j^i) - y^i)_1 \right| + 2^{j/2} \left| (\Phi_{\xi^i}(x^i, \xi_j^i) - y^i)_1^\dagger \right| \gtrsim 2^{j-k} \quad (7. 31)$$

for some $i \in \mathbf{U}$, provided that the implied constant in (7. 30) is sufficiently large. By inserting estimate (7. 31) into (7. 20), and carrying out the same estimation as in step 1, we obtain

$$\int_{{}^c\mathbf{B}_\delta^*} |\Omega_j(x, y)| dx \lesssim \frac{2^{-j}}{\delta}, \quad y \in \mathbf{B}_\delta. \quad (7. 32)$$

□

In order to show (7. 1), it is suffice to consider

$$\sum_{\mathbf{t}} \int |a(y)| \left\{ \int_{{}^c\mathbf{B}_\delta^*} |\Omega_{\mathbf{t}}(x, y)| dx \right\} dy. \quad (7. 33)$$

However, from (7. 5) in Lemma 7.1, the summation over all \mathbf{t} in (7. 33) will be dominated by

$$\int |a(y)| \left\{ \sum_{j=1}^{\infty} \int_{{}^c\mathbf{B}_\delta^*} |\Omega_j(x, y)| dx \right\} dy. \quad (7. 34)$$

For $2^j \leq \delta^{-1}$, we write

$$\int a(y) \Omega_j(x, y) dy = \int a(y) (\Omega_j(x, y) - \Omega_j(x, z)) dy \quad (7. 35)$$

as a fact of $\int a(y) dy = 0$. By (7. 6) in Lemma 7.1, we have

$$\int |a(y)| \left\{ \int |\Omega_j(x, y) - \Omega_j(x, 0)| dx \right\} dy \lesssim \|a\|_{\mathbf{L}^1} 2^j \delta \quad (7. 36)$$

provided that a is supported on \mathbf{B}_δ . By summing over all such j s, we have

$$\left(\sum_{2^j \leq \delta^{-1}} 2^j \right) \delta \leq \mathbf{const}. \quad (7. 37)$$

For $2^j > \delta^{-1}$, by (7. 7) in Lemma 7.1, we have

$$\int |a(y)| \left\{ \int_{\mathbf{B}_\delta^*} |\Omega_j(x, y)| dx \right\} dy \lesssim \|a\|_{L^1} \frac{2^{-j}}{\delta} \quad (7. 38)$$

for $2^j > \delta^{-1}$. By summing over all such j s, we have

$$\left(\sum_{2^j > \delta^{-1}} 2^{-j} \right) \delta^{-1} \leq \mathbf{const.} \quad (7. 39)$$

At this point, (7. 1) is proved. Together with (6. 6), we have \mathcal{F} is bounded from \mathbf{H}^1 to \mathbf{L}^1 . In the other hand, the adjoint operator of \mathcal{F} can be written as

$$\begin{aligned} (\mathcal{F}^* f)(x) &= \int f(y) \Omega^*(x, y) dy \\ &= \int f(y) \left\{ \int e^{2\pi i(\Phi(y, \xi) - x \cdot \xi)} \overline{\sigma}(y, \xi) d\xi \right\} dy. \end{aligned} \quad (7. 40)$$

Observe that its kernel Ω^* has x and y reversed in the role of Ω in (2. 7). As discussed in 4.8, chapter IX of [3], the *region of influence* associated to \mathcal{F}^* shall be defined as in the previous section, whereas \mathbf{B}_δ^* in (6. 17) is constructed in terms of $^*R_{S_i}^{V_i}$ as in (6. 18). By carrying out the same estimation developed in this section on Ω^* , we shall have (7. 1) also valid for \mathcal{F}^* .

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